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Differential equations satisfied by principal series Whittaker functions on $SU(2, 2)$

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Abstract

In this talk, we discuss about a holonomic system of differential equations for Whittaker functions associated with the principal series representation of $SU(2; 2)$ with higher dimensional minimal K -type.

1 Introduction

Throughout, let G be the simple real Lie group $SU(2, 2)$ of rank two, and let

$K = S(U(2) \times U(2))$: the maximal compact subgroup of G

π : an irreducible representation of G which is K -admissible.

For the representation π , there are two types of Whittaker model with respect to a character η of N (a spherical subgroup of G). One is the smooth model, and the other is the algebraic models induced by the space of algebraic Whittaker vectors:

$$W(\pi, \eta) := \text{Hom}_{(\mathfrak{g}, K)}(\pi|_K, C^\infty\text{-Ind}_N^G(\eta)),$$

Here, \mathfrak{g} is the Lie algebra of G , $\pi|_K$ is the subspace of K -finite vectors in π and $C^\infty\text{-Ind}_N^G(\eta)$ is the right G -module smoothly induced from η .

Our aim is a characterization of the space $W(\pi, \eta)$ for the principal series representation π of G associated with a minimal parabolic subgroup, which leads to a description of the following challenging question associated to π .

Question. For each intertwiner Φ in $W(\pi, \eta)$, what is the image of Φ ? Equivalently, for each K -type τ occurring in π , one can ask the image of the τ -isotypic component in π . The latter image is called the space of Whittaker functions of π with respect to τ .

The natural and classical approach. Let τ be a K -type belonging to π , and f_1, \dots, f_n be its a basis in π . Denote by $\phi_j(g)$ the image of f_j under a fixed intertwiner Φ . Then, for each j and k in K , the function $(k\phi_j)(g) = \phi_j(gk)$ is a linear combination of the functions $\phi_1(g), \dots, \phi_n(g)$. Thus, it is enough to consider the functions ϕ_j on A for our purpose.

Assume that \mathcal{C} is a square matrix of size $\dim(\tau)$, with entries in the universal enveloping algebra of \mathfrak{g} , so that

$$\pi(\mathcal{C}) \circ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \gamma \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \quad (1)$$

for some constant $\gamma \in \mathbb{C}$.

By applying Φ to the identity (1) we get the following system of differential equations (the A -radial part)

$$\mathcal{R}(\mathcal{C}) \circ \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix} = \gamma \cdot \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix}, \quad a \in A$$

where \mathcal{R} denotes the infinitesimal action of G on $C^\infty\text{-Ind}_N^G(\eta)$. Thus, one can regard the space $W(\pi, \eta)$ as a subset of the solution space $Sol(\mathcal{R}(\mathcal{C}))$ of the system by sending Φ to the functions $\{\phi_j(a)\}$.

Remark. Recall that Whittaker functions satisfy differential equations with regular singularities at "0". The most important requirements for choosing a basis for τ are the simplicity and symmetricity of the series expansion of these functions $\phi_j(a) (a \in A)$ around 0 and of the system of differential equations arising from (1).

Principal series $\pi_{s,\chi}$. Let

$$\mathfrak{a} = \{a(t_1, t_2) = \begin{pmatrix} 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & t_2 \\ t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R}\} \subset \mathfrak{g},$$

$$M = \{\text{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta})\} \oplus \{1_4, \text{diag}(1, -1, 1, -1)\}.$$

Define linear functions λ_1 and λ_2 on \mathfrak{a} by $\lambda_1(a(t_1, t_2)) = t_1$ and $\lambda_2(a(t_1, t_2)) = t_2$. Then the set $\{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$ forms the restricted root system of type C_2 for the pair $(\mathfrak{g}, \mathfrak{a})$. Define $\lambda_1 \pm \lambda_2, 2\lambda_1$ and $2\lambda_2$ to be positive. Let P_{min} be the minimal parabolic subgroup of G with Langlands decomposition $P_{min} = MAN$, where N is the unipotent subgroup defined by the root spaces corresponding to positive roots. For the character $s \otimes \chi$ of M , $s \in \mathbb{Z}$, and the \mathbb{C} -valued real linear form $\mu = \mu_1\lambda + \mu_2\lambda_2$, one has the principal series representation

$$\pi_{s,\chi} := \text{Ind}_P^G((s \otimes \chi)_M \otimes e^{\mu+\rho} \otimes 1_N),$$

where 1_N is the trivial character of N .

The main object in the paper is the 8-dimensional space $W(\pi_{s,\chi}, \eta)$ of algebraic Whittaker vectors (see Kostant [2]) for non-degenerate character η (unitary) of N . Note that it is sufficient for our purpose to assume that $s \geq 0$.

1.1 Some previous results

Let us recall some known identities as in (1) and previous results for the space $W(\pi, \eta)$. The first example is the classical **Casimir** equation: let Ω be the Casimir operator of G . Then we have the following identity

$$\pi_{s,\chi}(\Omega)v = \chi_{\pi_{s,\chi}}(\Omega)v,$$

where $\chi_{\pi_{s,\chi}}$ is the infinitesimal character and v is a differential vector. This identity gives us an injection of $W(\pi_{s,\chi}, \eta)$ into the solution space $Sol(\mathcal{R}(\Omega))$ of the above equation. Note that the space $Sol(\mathcal{R}(\Omega))$ is of infinite dimension.

Let π be a discrete series representation of G and τ be its minimal K -type. Then Yamashita [10] defined an operator $D_{\pi,\tau}$ on τ under π :

$$\pi(D_{\pi,\tau})\tau = 0.$$

This gives us an injection of $W(\pi, \eta)$ into the solution space $Sol(\mathcal{R}(D_{\pi,\tau}))$ of the operator $\mathcal{R}(D_{\pi,\tau})$. Moreover, under certain conditions, he showed that

$$W(\pi, \eta) \cong Sol(\mathcal{R}(D_{\pi,\tau}))$$

as vector spaces. This result is not just for the group G (see [10] and [11]).

Let π be the principal series representation of $G = Sp(2, \mathbb{R})$ as in [6], and τ be the minimal K -type of π . In [6], the authors obtained a matrix, of size $\dim(\tau)$, formula of the form $\pi(\mathcal{D})v = \gamma v$ which implies

$$W(\pi, \eta) \cong Sol(\mathcal{R}(\Omega), \mathcal{R}(\mathcal{D})),$$

where Ω stands for the Casimir operator of $Sp(2, \mathbb{R})$. Note that the possible value of $\dim(\tau)$ is 1 or 2. The degree of \mathcal{D} is 4 if $\dim(\tau) = 1$, and 2 for the case of dimension 2.

Remark. In the case $s = 0$ and $s = 1$, the corresponding spaces $W(\pi_{s,\chi}, \eta)$ behave quite similar to the above mentioned cases for $G = Sp(2, \mathbb{R})$, and are studied in [4].

2 Differential equations

We begin by providing some formulas for the multiplicity one K -types $\tau_{[0,s;l]}$ in the principal series $\pi_{s,\chi}$. These formulas come from the explicit (\mathfrak{g}, K) -module structure of $\pi_{s,\chi}$ which originally discussed by Oda [7].

Note that the space of the adjoint K -representation $(Ad, \mathfrak{p}_{\mathbb{C}})$ is generated by the matrix units E_{ij+2} and E_{i+2j} ($1 \leq i, j \leq 2$) and denote by \mathcal{E}_{ij+2} and \mathcal{E}_{i+2j} their infinitesimal actions with respect to π_s . Let denote $\mathbf{F}_{[s;l]}$ the transpose of the vector (f_0, f_1, \dots, f_s) , where $\{f_j : 0 \leq j \leq s\}$ is the "nice" basis of $\tau_{[0,s;l]}$ introduced in [1] and $\mathbf{c}_q := q/s$ for $0 \leq q \leq s$.

Formula 1. (*Casimir equation*) Let Ω be the Casimir operator. Then we have

$$\pi_{s,\chi}(\Omega) \cdot \mathbf{F}_{[s;l]} = (\mu_1^2 + \mu_2^2 + \frac{1}{2}s^2 - 10)\mathbf{F}_{[s;l]}.$$

Formula 2. (*Shift equations*) Set $\nu_1 = \frac{1}{2}(s+l)$ and $\nu_2 = \frac{1}{2}(s-l)$. Then we have

$$\pi_{s,\chi}(\bar{\mathcal{Q}}) \cdot \mathbf{F}_{[s;l]} = \frac{1}{4}(\mu_1^2 - (\nu_1 + 1)^2) \mathbf{F}_{[s;l]},$$

and

$$\pi_{s,\chi}(\mathcal{Q}) \cdot \mathbf{F}_{[s;l]} = \frac{1}{4}(\mu_2^2 - (\nu_2 - 1)^2) \mathbf{F}_{[s;l]},$$

where $\bar{\mathcal{Q}} = \{\bar{Q}_{ij}\}_{0 \leq i,j \leq s}$ and $\mathcal{Q} = \{Q_{ij}\}_{0 \leq i,j \leq s}$ are square matrices given by

$$\begin{aligned}\bar{Q}_{qq-1} &= -\mathbf{c}_q(\mathcal{E}_{24}\mathcal{E}_{32} + \mathcal{E}_{14}\mathcal{E}_{31}) \\ \bar{Q}_{qq+1} &= -(1 - \mathbf{c}_q)(\mathcal{E}_{23}\mathcal{E}_{42} + \mathcal{E}_{13}\mathcal{E}_{41}) \\ \bar{Q}_{qq} &= (1 - \mathbf{c}_q)(\mathcal{E}_{23}\mathcal{E}_{32} + \mathcal{E}_{13}\mathcal{E}_{31}) + \mathbf{c}_q(\mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42})\end{aligned}$$

and

$$\begin{aligned}Q_{qq-1} &= \mathbf{c}_q(\mathcal{E}_{32}\mathcal{E}_{24} + \mathcal{E}_{31}\mathcal{E}_{14}) \\ Q_{qq+1} &= (1 - \mathbf{c}_q)(\mathcal{E}_{42}\mathcal{E}_{23} + \mathcal{E}_{41}\mathcal{E}_{13}) \\ Q_{qq} &= \mathbf{c}_q(\mathcal{E}_{32}\mathcal{E}_{23} + \mathcal{E}_{31}\mathcal{E}_{13}) + (1 - \mathbf{c}_q)(\mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24})\end{aligned}$$

for $0 \leq q \leq s$, but all other entries are 0.

Formula 3. (*Annihilation equations*) We have

$$\pi_{s,\chi}(\mathcal{A}) \cdot \mathbf{F}_{[s;l]} = 0,$$

and

$$\pi_{s,\chi}(\bar{\mathcal{A}}) \cdot \mathbf{F}_{[s;l]} = 0,$$

where $\mathcal{A} = \{A_{ij}\}$ and $\bar{\mathcal{A}} = \{\bar{A}_{ij}\}$ are square matrix whose non-zero entries are given by

$$\begin{aligned}A_{jj-1} &= -\mathcal{E}_{31}\mathcal{E}_{14} - \mathcal{E}_{32}\mathcal{E}_{24}, \\ A_{jj} &= \mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24} - \mathcal{E}_{31}\mathcal{E}_{13} - \mathcal{E}_{32}\mathcal{E}_{23}, \\ A_{jj+1} &= \mathcal{E}_{41}\mathcal{E}_{13} + \mathcal{E}_{42}\mathcal{E}_{23},\end{aligned}$$

and

$$\begin{aligned}\bar{A}_{jj-1} &= -\mathcal{E}_{14}\mathcal{E}_{31} - \mathcal{E}_{24}\mathcal{E}_{32}, \\ \bar{A}_{jj} &= \mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42} - \mathcal{E}_{13}\mathcal{E}_{31} - \mathcal{E}_{23}\mathcal{E}_{32}, \\ \bar{A}_{jj+1} &= \mathcal{E}_{13}\mathcal{E}_{41} + \mathcal{E}_{23}\mathcal{E}_{42},\end{aligned}$$

for $1 \leq j \leq s-1$.

Proposition 2.1. On the K -type $\tau_{[0,s;l]}$ with respect to the action $\pi_{s,\chi}$ we have

$$\mathcal{Q} + \bar{\mathcal{Q}} = \Omega/4.$$

2.1 A holonomic system of rank 8

Coordinate system. Since the \mathbb{R} -split torus A for our case is two dimensional, one may choose the coordinate system (y_1, y_2) . Denote the Euler operators $y_1 \frac{\partial}{\partial y_1}$ and $y_2 \frac{\partial}{\partial y_2}$ with respect to this system by ∂_1 and ∂_2 , respectively.

We now define the matrix differential operator $\bar{\mathcal{D}}$ by

$$\begin{pmatrix} \bar{d}_{00} & \bar{d}_{01} & 0 & \cdots & 0 \\ 0 & \bar{d}_{11} & \bar{d}_{12} & \cdots & 0 \\ 0 & 0 & \bar{d}_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & \bar{d}_{s-2s-2} & \bar{d}_{s-2s-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_{s-1s-1} & \bar{d}_{s-1s} \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_{ss-1} & \bar{d}_{ss} \end{pmatrix}$$

where

$$d_{qq} = \frac{1}{4}((\partial_1 - q)^2 - \mu_1^2) - \xi \bar{\xi} y_1^2, \quad d_{qq+1} = \bar{\xi} y_1(\partial_2 + \frac{1}{2}s - q) + \bar{\xi} y_1 y_2$$

for $q = 0, \dots, s-1$ and

$$d_{ss} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_1^2) - \xi \bar{\xi} y_1^2 - y_2^2 - \nu_1 y_2$$

$$d_{ss-1} = -\xi y_1(\partial_2 + \frac{1}{2}s) + \xi y_1 y_2.$$

We also define the matrix differential operator \mathcal{D} by

$$\begin{pmatrix} d_{00} & d_{01} & 0 & \cdots & 0 \\ d_{10} & d_{11} & 0 & \cdots & 0 \\ 0 & a_{32} & d_{33} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & d_{s-2s-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & d_{s-1s-2} & d_{s-1s-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_{ss-1} & d_{ss} \end{pmatrix}$$

where

$$d_{00} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \bar{\xi} y_1^2 - y_2^2 - \nu_2 y_2$$

$$d_{01} = -\bar{\xi} y_1(\partial_2 - \frac{1}{2}s) - \bar{\xi} y_1 y_2$$

and

$$d_{qq} = \frac{1}{4}((\partial_1 - s + q)^2 - \mu_2^2) - \xi \bar{\xi} y_1^2, \quad d_{qq-1} = \xi y_1(\partial_2 + q - \frac{1}{2}s) - \xi y_1 y_2$$

for $q = 1, \dots, s$. Here, the parameters ξ and $\bar{\xi}$ are associated to the character η .

By using Formulas 2 and 3, one can see that the Whittaker functions of $\pi_{s,\chi}$ with respect to $\tau_{[0,s;l]}$ satisfy the system of differential equations $\mathcal{D} = 0$ and $\bar{\mathcal{D}} = 0$. Moreover, we have the following result which characterizes the Whittaker functions of $\pi_{s,\chi}$ with respect to $\tau_{[0,s;l]}$.

Theorem 2.2. *For $s \geq 2$, the natural map from $W(\pi_{s,\chi}, \eta)$ into $\text{Ker}(\bar{\mathcal{D}}, \mathcal{D})$ is bijection if $\pi_{s,\chi}$ is irreducible and η is a nondegenerate unitary character of N .*

Here, we also have the following formula in the case $s = 0$, which is analogue to the class one case for $Sp(2, \mathbb{R})$ in [5]. Write W for the little Weyl group for $(\mathfrak{g}, \mathfrak{a})$, and (ρ_1, ρ_2) for the pair $(3, 2)$ related to the half sum.

Theorem 2.3. *Let $\pi_{0,\chi}$ be an irreducible principal series with parameter $\mu = (\mu_1, \mu_2) \in \mathfrak{a}_{\mathbb{C}}^*$, and set $\varepsilon = \frac{1-\chi(-1)}{2}$. Then the function ϕ_μ on A defined by*

$$\begin{aligned} \phi_\mu(y_1, y_2) = & y_1^{\rho_1} y_2^{\rho_2} \sum_{m,n \geq 0} \frac{U_{m,n}^0}{2^{2n} \left(\frac{\mu_1 - \varepsilon}{2} + 1\right)_m \left(\frac{\mu_2 - \varepsilon}{2} + 1\right)_n} \times y_1^{\mu_1 + 2m} y_2^{\frac{\mu_1 + \mu_2}{2} + 2n} \\ & + \frac{\varepsilon U_{m,n}^1}{2^{2n+1} \left(\frac{\mu_1 - \varepsilon}{2} + 1\right)_m \left(\frac{\mu_2 - \varepsilon}{2} + 1\right)_{n+1}} \times y_1^{\mu_1 + 2m} y_2^{\frac{\mu_1 + \mu_2}{2} + 2n+1}, \end{aligned}$$

is a Whittaker function, on A , of $\pi_{0,\chi}$ with the K -type $\tau_{[0,0;2\varepsilon]}$. Moreover, the intertwiners $\Phi_{\omega(\mu)}$ attached to the function $\phi_{\omega(\mu)}(y_1, y_2)$ form a basis of the 8-dimensional space $W(\pi_{0,\chi}, \eta)$. Here,

$$U_{m,n}^t := \sum_{j=0}^{\min(m,n)} \frac{\left(\frac{\mu_1 - \varepsilon}{2} + n + 1 + t\right)_{m-j}}{(m-j)!(n-j)!j! \left(\frac{\mu_1 + \mu_2}{2} + 1\right)_j \left(\frac{\mu_1 - \mu_2}{2} + 1\right)_{m-j}}$$

for $t = 0, 1$.

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